NORTH-HOLLAND

Inequalities for Permanents of Hermitian Matrices

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ABSTRACT

For given integers n_1, $n_2 \geq 1$, we consider two hermitian matrices of order $n = n_1 + n_2$, written in block form:

$$A = \begin{pmatrix} A_{11} & C_{12} \\ C_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & C_{12} \\ C_{21} & B_{22} \end{pmatrix},$$

where every matrix X_{ij} has dimension $n_i \times n_j$, $1 \leq i$, $j \leq 2$. It is proved that if $A_{11} \geq B_{11} \geq 0$, $A_{22} \geq B_{22} \geq 0$, then per $A \geq$ per $B \geq 0$. © 1997 Elsevier Science Inc.

All matrices considered in this paper have entries belonging to the complex field C.

Let $A = (a_{ij})$ be a matrix of order n, and let per A denote the permanent of A; that is,

$$\operatorname{per} A = \sum_{\sigma \in S_n} a_{1\sigma_1} \cdots a_{n\sigma_n},$$

where S_n is the set of all $n!$ permutations $\sigma = (\sigma_1, \dots, \sigma_n)$ of the set $\{1, \dots, n\}$. Let $A = BC$, where $B = (b_{ik})$ and $C = (c_{kj})$ are respectively $n \times m$ and $m \times n$ matrices, $n \leq m$. By the Binet-Cauchy formula for

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permanents,

$$\operatorname{per} A = \frac{1}{n!} \sum_{1 \le k_1, \dots, k_n \le m} \operatorname{per}\begin{pmatrix} b_{1k_1} & \cdots & b_{1k_n} \\ \vdots & & \vdots \\ b_{nk_1} & \cdots & b_{nk_n} \end{pmatrix} \operatorname{per}\begin{pmatrix} c_{k_1 1} & \cdots & c_{k_1 n} \\ \vdots & & \vdots \\ c_{k_n 1} & \cdots & c_{k_n n} \end{pmatrix}.$$

Let $S(n)$ be the set of hermitian matrices of order n, and let $S^+(n)$ be the set of positive semidefinite matrices. We write $A \ge B$ for $A, B \in S(n)$ if $A - B \in S^+(n)$.

EXAMPLE 1. Let $A = (a_{ij}) \in S^+(n)$, let $x = (x_1, \dots, x_n) \in C^n$ be any (variable) vector, and let $y = (y_1, \dots, y_n) \in C^n$ be any (fixed) vector. By the Cauchy-Schwarz inequality

$$\left(\sum_{1 \le i, j \le n} a_{ij} x_i \bar{x}_j \right) \left(\sum_{1 \le i, j \le n} a_{ij} y_i \bar{y}_j \right)$$
$$\ge \left(\sum_{1 \le i, j \le n} a_{ij} x_i \bar{y}_j \right) \left(\overline{\sum_{1 \le i, j \le n} a_{ij} x_i \bar{y}_j} \right) \ge 0.$$

Set

$$b_i = \sum_{1 \le j \le n} a_{ij} \bar{y}_j, \qquad 1 \le i \le n,$$

$$b = \sum_{1 \le i, j \le n} a_{ij} y_i \bar{y}_j.$$

Therefore, $b \ge 0$, and if $b > 0$, we obtain

$$\sum_{1 \le i, j \le n} a_{ij} x_i \bar{x}_j \ge \frac{\left(\sum_{1 \le i \le n} b_i x_i \right) \left(\overline{\sum_{1 \le i \le n} b_i x_i} \right)}{b} = \sum_{1 \le i, j \le n} \frac{b_i \bar{b}_j}{b} x_i \bar{x}_j \ge 0.$$

So $A \ge B \ge 0$, where

$$B = \left(\frac{b_i \bar{b}_j}{b} \right)$$

is a hermitian matrix of rank 1.

The following representation of matrices belonging to $S^+(n)$ is often applied. Let $A \in S^+(n)$, and let rank $A \leq m$. Then there exists an $n \times m$ matrix C such that $A = CC^*$.

Now let $1 \leq r \leq n$, H, $K \subset \{1, \ldots, n\}$, $|H| = |K| = r$. We denote by $A(H, K)$ the corresponding submatrix of order r of the matrix A. Then

$$A(H, K) = C(H, I_m)C^*(I_m, K),$$

where $I_m = \{1, \ldots, m\}$. By the Binet-Cauchy formula

$$\operatorname{per} A(H, K) = \frac{1}{r!} \sum_{1 \leq i_1, \ldots, i_r \leq m} \operatorname{per} C(H; i_1, \ldots, i_r) \operatorname{per} C^*(i_1, \ldots, i_r; K).$$

Thus,

$$\operatorname{per} A(H, K) = \frac{1}{r!} \sum_{1 \leq i_1, \ldots, i_r \leq m} \operatorname{per} C(H; i_1, \ldots, i_r) \overline{\operatorname{per} C(K; i_1, \ldots, i_r)}.$$

LEMMA. *Let $A \in S^+(n)$, $B \in S^+(n)$, $A \geq B$, and let r be a given positive integer, $1 \leq r \leq n$. Then there exist finite sets L, M, $L \supset M$, and a family $(c_\lambda)_{\lambda \in L}$ of complex valued functions defined on the set*

$$V = \{H \subset \{1, \ldots, n\} \mid |H| = r\}$$

such that for all H, $K \in V$ the equalities

$$\operatorname{per} A(H, K) = \sum_{\lambda \in L} c_\lambda(H) \overline{c_\lambda(K)},$$

$$\operatorname{per} B(H, K) = \sum_{\lambda \in M} c_\lambda(H) \overline{c_\lambda(K)}$$

hold.

Proof. For the matrices B, $A - B \in S^+(n)$ there exist matrices C_1, C_2 of order n such that $B = C_1C_1^*$, $A - B = C_2C_2^*$, and thus $A = C_1C_1^* +$

$C_2 C_2^*$. Let

$$C_1 = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}, \qquad C_2 = \begin{pmatrix} c_{1,n+1} & \cdots & c_{1,2n} \\ \vdots & & \vdots \\ c_{n,n+1} & \cdots & c_{n,2n} \end{pmatrix},$$

and set

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1n} & c_{1,n+1} & \cdots & c_{1,2n} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} & c_{n,n+1} & \cdots & c_{n,2n} \end{pmatrix}.$$

Then $A = CC^*$. Therefore, for any $H,\ K \in V$ we have

$$\operatorname{per} A(H, K) = \frac{1}{r!} \sum_{1 \le i_1, \dots, i_r \le 2n} \operatorname{per} C(H; i_1, \dots, i_r) \overline{\operatorname{per} C(K; i_1, \dots, i_r)},$$

$$\operatorname{per} B(H, K) = \frac{1}{r!} \sum_{1 \le i_1, \dots, i_r \le n} \operatorname{per} C(H; i_1, \dots, i_r) \overline{\operatorname{per} C(K; i_1, \dots, i_r)}.$$

The second equality follows from the fact that the first n columns of the matrix C coincide with the corresponding columns of the matrix C_1.

Now let $L = \{1, \dots, 2n\}^r$, $M = \{1, \dots, n\}^r$; then for $\lambda = (i_1, \dots, i_r) \in L$, $H \in V$ set

$$c_\lambda(H) = \frac{1}{\sqrt{r!}} \operatorname{per} C(H; i_1, \dots, i_r).$$

The lemma is proved. ■

Obviously, the lemma is true also for $r = 0$. We now present an example of application of the lemma.

EXAMPLE 2. Let $m \ge 1$, n, r, $1 \le r \le n$, be given integers, and denote

$$V = \{H \subset \{1, \dots, n\} \mid |H| = r\}.$$

For $A_1, \dots, A_m \in S^+(n)$ set

$$F(A_1, \dots, A_m) = \sum_{(H, K) \in V \times V} \operatorname{per} A_1(H, K) \cdots \operatorname{per} A_m(H, K).$$

Now, if $A_1, \dots, A_m \in S^+(n)$, $B_1, \dots, B_m \in S^+(n)$ satisfy $A_1 \geq B_1, \dots, A_m \geq B_m$, then

$$F(A_1, \dots, A_m) \geq F(B_1, \dots, B_m).$$

Indeed, by the lemma, for every i, $1 \leq i \leq m$, there exist finite sets L_i, M_i, $L_i \supset M_i$, and a family of functions $(c_{\lambda_i}^{(i)})_{\lambda_i \in L_i}$ where $c_{\lambda_i}^{(i)} : V \to C$, such that

$$\operatorname{per} A_i(H, K) = \sum_{\lambda_i \in L_i} c_{\lambda_i}^{(i)}(H)\,\overline{c_{\lambda_i}^{(i)}(K)},$$

$$\operatorname{per} B_i(H, K) = \sum_{\lambda_i \in M_i} c_{\lambda_i}^{(i)}(H)\,\overline{c_{\lambda_i}^{(i)}(K)}$$

for all $(H, K) \in V \times V$.

Now we consider the sets $L = L_1 \times \cdots \times L_m$, $M = M_1 \times \cdots \times M_m$. Obviously, $L \supset M$. For $\lambda = (\lambda_1, \dots, \lambda_m) \in L$ we set

$$c_\lambda = c_{\lambda_1}^{(1)} \cdots c_{\lambda_m}^{(m)} : V \to C.$$

We have

$$F(A_1, \dots, A_m) = \sum_{(H, K) \in V \times V} \sum_{\lambda \in L} c_\lambda(H)\,\overline{c_\lambda(K)},$$

$$F(B_1, \dots, B_m) = \sum_{(H, K) \in V \times V} \sum_{\lambda \in M} c_\lambda(H)\,\overline{c_\lambda(K)},$$

$$\begin{aligned} F(A_1, \dots, A_m) - F(B_1, \dots, B_m) &= \sum_{(H, K) \in V \times V} \sum_{\lambda \in L \setminus M} c_\lambda(H)\,\overline{c_\lambda(K)}, \\ &= \sum_{\lambda \in L \setminus M} \sum_{(H, K) \in V \times V} c_\lambda(H)\,\overline{c_\lambda(K)}, \\ &= \sum_{\lambda \in L \setminus M} \left| \sum_{H \in V} c_\lambda(H) \right|^2 \geq 0. \end{aligned}$$

Our assertion is thus proved.

We note that if $A \in S^+(n)$, $B \in S^+(n)$, $A \geq B$, then $\bar{A} \in S^+(n)$, $\bar{B} \in S^+(n)$, $\bar{A} \geq \bar{B}$. Therefore, we can obtain from the inequality,

$$F(A_1, \dots, A_m) \geq F(B_1, \dots, B_m)$$

similar inequalities, in which some of the pairs (A_i, B_i) are replaced by $(\bar{A}_i, \bar{B}_i)$. For instance, for $m = 3$, we have

$$F\left(A_1, \bar{A}_2, A_3\right) \geq F\left(B_1, \bar{B}_2, B_3\right).$$

Note that from Example 1 [for $y = (1, \dots, 1) \in C^n$] and Example 2 (for $m = 1$) follows the truth of the conjecture in the paper [1] (Problem 1), which is proved in [2] (Theorem 3).

We now consider block matrices. For given integers n_1, $n_2 \geq 1$, let X be a matrix of order $n = n_1 + n_2$, written in block form:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

where the matrix X_{ij} has dimension $n_i \times n_j$, $1 \leq i$, $j \leq 2$. Let $I_1 = \{1, \dots, n_1\}$, $I_2 = \{n_1 + 1, \dots, n_1 + n_2\}$, $I = I_1 \cup I_2 = \{1, \dots, n\}$. Then, for H, $K \subset I$, $|H| = |K|$, let

$$p(H, K) = \operatorname{per} X(H, K).$$

Using the Laplace theorem, we expand $\operatorname{per} X$ relative to the first n_1 rows of X:

$$\operatorname{per} X = \sum_{H \subset I, |H| = n_1} p(I_1, H) p(I_2, I \setminus H). \tag{1}$$

For a fixed set $H \subset I$, $|H| = n_1$, let $i_1 = |H \cap I_1|$. Then $|H \cap I_2| = n_1 - i_1$. Expanding $p(I_1, H)$ relative to the i_1 columns of $X(I_1, H)$, which belong to I_1, we obtain

$$p(I_1, H) = \sum_{J_1 \subset I_1, |J_1| = i_1} p(J_1, H \cap I_1) p(I_1 \setminus J_1, H \cap I_2).$$

We have

$$|(I\setminus H)\cap I_1| = |I_1\setminus H\cap I_1| = n_1 - i_1,$$

$$|(I\setminus H)\cap I_2| = |I_2\setminus H\cap I_2| = n_2 - n_1 + i_1.$$

Expanding $p(I_2, I\setminus H)$ relative to the $n_2 - n_1 + i_1$ columns of $X(I_2, I\setminus H)$, which belong to I_2, we obtain

$$p(I_2, I\setminus H) = \sum_{J_2\subset I_2,\, |J_2| = n_2 - n_1 + i_1} p(J_2, I_2\setminus H\cap I_2)\, p(I_2\setminus J_2, I_1\setminus H\cap I_1).$$

Let $H_1 = H\cap I_1$, $H_2 = I_2\setminus H\cap I_2$. Then

$$p(I_1, H)p(I_2, I\setminus H) = \sum_{\substack{J_1\subset I_1,\, J_2\subset I_2\\ |J_1|=|H_1|,\, |J_2|=|H_2|}} p(J_1, H_1)p(I_1\setminus J_1, I_2\setminus H_2)p(J_2, H_2) \times p(I_2\setminus J_2, I_1\setminus H_1).$$

We return to the set $H\subset I$, $|H| = n_1$. The choice of such a set is equivalent to the choice of the pair of sets $(H\cap I_1, H\cap I_2)$. Obviously, $H\cap I_1\subset I_1$, $H\cap I_2\subset I_2$, $|H\cap I_1| + |H\cap I_2| = n_1$. In our notation

$$H\cap I_1 = H_1, \qquad H\cap I_2 = I_2\setminus H_2,$$

so that $|H_1| + n_2 - |H_2| = n_1$, $|H_1| - |H_2| = n_1 - n_2$. Thus, (1) can be written in the form

$$\operatorname{per} X = \sum_{\substack{H_1\subset I_1,\, H_2\subset I_2\\ |H_1|-|H_2| = n_1 - n_2\\ J_1\subset I_1,\, J_2\subset I_2\\ |J_1|=|H_1|,\, |J_2|=|H_2|}} p(J_1, H_1)p(J_2, H_2)p(I_1\setminus J_1, I_2\setminus H_2) \times p(I_2\setminus J_2, I_1\setminus H_1). \qquad (2)$$

We now take $t \in C$ and consider the matrix

$$X(t) = \begin{pmatrix} X_{11} & tX_{12} \\ \bar{t}X_{21} & X_{22} \end{pmatrix}.$$

Then $\operatorname{per} X(t)$ is a polynomial in $t, \bar{t}$. Calculating $\operatorname{per} X(t)$ by (2), we consider, in the sum, the term corresponding to a given (H_1, H_2, J_1, J_2). Its degree in t is equal to $n_1 - |H_1| = n_2 - |H_2|$, and its degree in $\bar{t}$ is equal to $n_2 - |H_2| = n_1 - |H_1|$, i.e., the degrees in t and $\bar{t}$ are equal. Let $k = n_1 - |H_1| = n_2 - |H_2|$ be the common degree. Then $|H_1| = n_1 - k$, $|H_2| = n_2 - k$; therefore $0 \leq k \leq \min\{n_1, n_2\} = m$. Thus,

$$\operatorname{per} X(t) = \sum_{0 \leq k \leq m} p_k(X)|t|^{2k}, \qquad t \in C.$$

For all k, $0 \leq k \leq m$, we have

$$p_k(X) = \sum_{\substack{H_1 \subset I_1,\, H_2 \subset I_2 \\ J_1 \subset I_1,\, J_2 \subset I_2 \\ |H_1| = |J_1| = n_1 - k \\ |H_2| = |J_2| = n_2 - k}} p(J_1, H_1) P(J_2, H_2) p(I_1 \setminus J_1, I_2 \setminus H_2)$$

$$\times\, p(I_2 \setminus J_2, I_1 \setminus H_1). \tag{3}$$

In particular, $p_0(X) = \operatorname{per} X_{11} \operatorname{per} X_{22}$.

We apply (3) to obtain a result about $p_k(A)$, $p_k(B)$ for two hermitian matrices of the form

$$A = \begin{pmatrix} A_{11} & C_{12} \\ C_{21} & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & C_{12} \\ C_{21} & B_{22} \end{pmatrix}.$$

THEOREM. *Assume that*

$$A_{11} \in S^+(n_1), \qquad B_{11} \in S^+(n_1), \qquad A_{11} \geq B_{11},$$
$$A_{22} \in S^+(n_2), \qquad B_{22} \in S^+(n_2), \qquad A_{22} \geq B_{22},$$
$$C_{12}^* = C_{21}.$$

Then $p_k(A) \geq p_k(B) \geq 0$ *for all* k, $0 \leq k \leq \min\{n_1, n_2\}$.

Proof. Note that, by our assumptions, the matrices A, B are hermitian [but, certainly, one cannot assert that $A, B \in S^{+}(n)$]. We fix k, $0 \leq k \leq \min\{n_1, n_2\}$.

For $i = 1, 2$, we apply the lemma (for $r = n_i - k$) to the matrices

$$A_{ii} \in S^{+}(n_i), \quad B_{ii} \in S^{+}(n_i), \qquad A_{ii} \geq B_{ii}.$$

It follows that there exist finite sets L_i, M_i, $L_i \supset M_i$, and a family

$$\left(c_{\lambda_i}^{(i)}\right)_{\lambda_i \in L_i}$$

of complex valued functions on the set

$$V_i = \left\{H_i \subset I_i \,\middle|\, |H_i| = n_i - k\right\},$$

such that

$$\operatorname{per} A_{ii}(J_i, H_i) = \sum_{\lambda_i \in L_i} c_{\lambda_i}^{(i)}(J_i)\,\overline{c_{\lambda_i}^{(i)}(H_i)},$$

$$\operatorname{per} B_{ii}(J_i, H_i) = \sum_{\lambda_i \in M_i} c_{\lambda_i}^{(i)}(J_i)\,\overline{c_{\lambda_i}^{(i)}(H_i)}$$

for all $(J_i, H_i) \in V_i \times V_i$. Hence

$$\operatorname{per} A_{11}(J_1, H_1) \operatorname{per} A_{22}(J_2, H_2)$$
$$= \sum_{(\lambda_1, \lambda_2) \in L_1 \times L_2} c_{\lambda_1}^{(1)}(J_1) c_{\lambda_2}^{(2)}(J_2)\,\overline{c_{\lambda_1}^{(1)}(H_1) c_{\lambda_2}^{(2)}(H_2)}, \tag{4}$$

$$\operatorname{per} B_{11}(J_1, H_1) \operatorname{per} B_{22}(J_2, H_2)$$
$$= \sum_{(\lambda_1, \lambda_2) \in M_1 \times M_2} c_{\lambda_1}^{(1)}(J_1) c_{\lambda_2}^{(2)}(J_2)\,\overline{c_{\lambda_1}^{(1)}(H_1) c_{\lambda_2}^{(2)}(H_2)} \tag{5}$$

for all $(J_1, H_1) \in V_1 \times V_1$, $(J_2, H_2) \in V_2 \times V_2$.

Define $L = L_1 \times L_2$ and $M = M_1 \times M_2$. We have $L \supset M$. We also define the family $(c_\lambda)_{\lambda \in L}$ of functions by

$$c_\lambda(H_1, H_2) = c^{(1)}_{\lambda_1}(H_1)\,\overline{c^{(2)}_{\lambda_2}(H_2)}$$

for all $\lambda = (\lambda_1, \lambda_2) \in L$, $(H_1, H_2) \in V_1 \times V_2$. In terms of this notation, (4) and (5) can be written in the form

$$\operatorname{per} A_{11}(J_1, H_1) \operatorname{per} A_{22}(J_2, H_2) = \sum_{\lambda \in L} c_\lambda(J_1, H_2)\,\overline{c_\lambda(H_1, J_2)}\,, \tag{6}$$

$$\operatorname{per} B_{11}(J_1, H_1) \operatorname{per} B_{22}(J_2, H_2) = \sum_{\lambda \in M} c_\lambda(J_1, H_2)\,\overline{c_\lambda(H_1, J_2)} \tag{7}$$

for all $(J_1, H_1) \in V_1 \times V_1$, $(J_2, H_2) \in V_2 \times V_2$.

For $(H_1, H_2) \in V_1 \times V_2$ we set

$$d(H_1, H_2) = \operatorname{per} A(I_1 \setminus H_1, I_2 \setminus H_2),$$

$$d^*(H_1, H_2) = \operatorname{per} A(I_2 \setminus H_2, I_1 \setminus H_1).$$

Since $C_{21} = C^*_{12}$, we have $d^*(H_1, H_2) = \overline{d(H_1, H_2)}$ for all $(H_1, H_2) \in V_1 \times V_2$.

Let $V = V_1 \times V_1 \times V_2 \times V_2$. From (3), (6) and (7) it follows that

$$p_k(A) = \sum_{(H_1, J_1, H_2, J_2) \in V} \left(\sum_{\lambda \in L} c_\lambda(J_1, H_2)\,\overline{c_\lambda(H_1, J_2)} \right) d(J_1, H_2) d^*(H_1, J_2) \tag{8}$$

and

$$p_k(B) = \sum_{(H_1, J_1, H_2, J_2) \in V} \left(\sum_{\lambda \in M} c_\lambda(J_1, H_2)\,\overline{c_\lambda(H_1, J_2)} \right) \times d(J_1, H_2) d^*(H_1, J_2). \tag{9}$$

For $\lambda \in L$ we set

$$G(\lambda) = \sum_{(H_1, J_1, H_2, J_2) \in V} c_\lambda(J_1, H_2)\overline{c_\lambda(H_1, J_2)}\, d(J_1, H_2) d^*(H_1, J_2).$$

For $(H_1, J_2) \in V_1 \times V_2$ we have $d^*(H_1, J_2) = \overline{d(H_1, J_2)}$. Therefore,

$$\begin{aligned} G(\lambda) &= \sum_{(H_1, J_1, H_2, J_2) \in V_1 \times V_1 \times V_2 \times V_2} c_\lambda(J_1, H_2) \\ &\quad \times d(J_1, H_2)\overline{c_\lambda(H_1, J_2) d(H_1, J_2)} \\ &= \sum_{(J_1, H_2) \in V_1 \times V_2} c_\lambda(J_1, H_2) d(J_1, H_2) \\ &\quad \cdot \sum_{(H_1, J_2) \in V_1 \times V_2} \overline{c_\lambda(H_1, J_2) d(H_1, J_2)} \\ &= \left| \sum_{(J_1, H_2) \in V_1 \times V_2} c_\lambda(J_1, H_2) d(J_1, H_2) \right|^2 \geq 0 \end{aligned}$$

for all $\lambda \in L$.

Changing the order of the summation in (8) and (9), we obtain

$$p_k(A) = \sum_{\lambda \in L} G(\lambda), \qquad p_k(B) = \sum_{\lambda \in M} G(\lambda).$$

Also, $L \supset M$; thus $p_k(A) \geq p_k(B) \geq 0$.

The theorem thus is proved. ∎

Corollary. *Assume that the conditions of the theorem hold. Then*

$$\operatorname{per} A \geq \operatorname{per} B \geq 0.$$

Indeed, this follows from the equalities

$$\operatorname{per} A = \sum_{0 \leq k \leq \min\{n_1, n_2\}} p_k(A), \qquad \operatorname{per} B = \sum_{0 \leq k \leq \min\{n_1, n_2\}} p_k(B).$$

The example of the hermitian matrices

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$$

shows that the natural generalization of the theorem to the case of three blocks ($n_1 = 2$, $n_2 = 1$, $n_3 = 1$) is not possible, because

$$\operatorname{per} A = -2, \qquad \operatorname{per} B = 0.$$

I got the idea of writing this paper after reading the interesting work [1], in which similar questions were discussed.

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